Hyperbolic Carathéodory conjecture

Valentin Ovsienko and Serge Tabachnikov

Dedicated to Vladimir Igorevich Arnold
in occasion of his 70th anniversary

Abstract

A quadratic point on a surface in $\mathbb{RP}^3$ is a point at which the surface can be approximated by a quadric abnormally well (up to order 3). We conjecture that the least number of quadratic points on a generic compact non-degenerate hyperbolic surface is 8; the relation between this and the classic Carathéodory conjecture is similar to the relation between the six-vertex and the four-vertex theorems on plane curves. Examples of quartic perturbations of the standard hyperboloid confirm our conjecture. Our main result is a linearization and reformulation of the problem in the framework of 2-dimensional Sturm theory; we also define a signature of a quadratic point and calculate local normal forms recovering and generalizing Tresse-Wilczynski's theorem.

Mathematical subject classification: 53A20, 53C99, 58K50.

1 Introduction

Almost one hundred years ago, S. Muchopadhyaya discovered two theorems on plane ovals (an oval is a smooth closed strictly convex plane curve). The first one is known as the four-vertex theorem: the curvature of a plane oval has at least 4 critical points. These critical points are the points at which the osculating circles are hyperosculating, that is, are third-order tangent to the curve. The second theorem concerns osculating conics and states that a smooth convex closed curve has at least 6 distinct points at which the osculating conics are hyperosculating. Such points are called sextactic. A smooth plane curve can be approximated by a conic at every point up to order 4; a point is sextactic if the order of approximation at this point is higher.

The four- and six-vertex theorems and their ramifications continue to attract interest, in great part due to work of V. I. Arnold who placed the subject into the framework of symplectic and contact topology [1, 2]. There is a wealth of new results in this field, see [10] for a survey.

It is natural to expect that there exist multi-dimensional versions of four- and six-vertex theorems but, so far, only the very first steps have been made in this direction [3, 4, 11, 16].

We consider the classical Carathéodory conjecture as belonging to the area. This conjecture states that a sufficiently smooth convex closed surface in $\mathbb{R}^3$ has at least 2 distinct umbilic points, that is, the points where the two principal curvatures are equal (see, e.g., [10] and references therein for a long and convoluted history of the subject). Umbilic points are analogs of vertices of plane curves: these are the points at which a sphere is abnormally (second-order) tangent to the surface. Let us note that a generic closed surface, even an immersed one, carries at least 4 umbilic points.

A smooth hypersurface M in $\mathbb{RP}^3$ can be approximated by a quadric at every point up to order 2. A point $x \in M$ is called *quadratic* if M can be approximated by a quadric at x up to order 3.[1] We view quadratic points of surfaces as 2-dimensional analogs of sextactic points. Quadratic points were studied in the classical literature, see [18, 14, 7], but we are aware of only one existence result: if a generic smooth surface in $\mathbb{RP}^3$ contains a hyperbolic disc, bounded by a Jordan parabolic curve, then there exists an odd number of quadratic points inside this disc (and hence, at least one) [16].

We assume that M is an orientable *non-degenerate hyperbolic* surface: the second quadratic form is non-degenerate and indefinite everywhere. Clearly, M is diffeomorphic to the 2-torus: $M \cong \mathbb{T}^2$. Indeed, at each point of M one has two *asymptotic directions*, the light-cone of the second quadratic form (equivalently defined as the tangent lines to the intersection of M with its tangent plane, see figure 1). It follows that the Euler characteristic of M is zero. The integral curves corresponding to the asymptotic directions are called *asymptotic lines* and they form a 2-web on M.

- *How many quadratic points are there on M?*

[1] Quadratic points are also called *hyperbonodes*, see [16]; in [11] these points are called *special*.

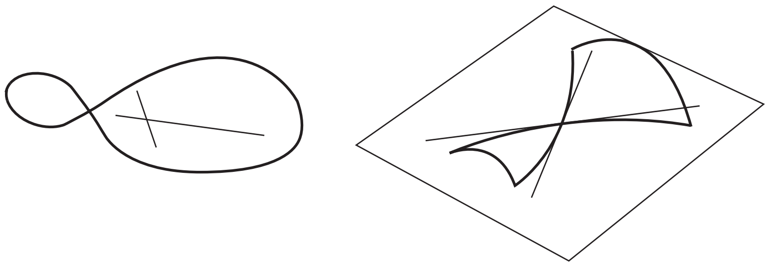

Figure 1: Non-degenerate hyperbolic surface and its asymptotic directions at a generic point.

We start with local analysis and define a *signature* of a non-gedenerate quadratic point: (s_1, s_2), with $s_i = \pm 1$, this is a $\mathrm{PGL}(4,\mathbb{R})$-invariant. Explicit formulæ for *normal forms* provide further differential invariants. This problem goes back to Tresse and Wilczynski, we give here a short proof of their classical result. We then calculate normal forms at the degeneration stratas that have been studied in [12, 8].

An example of a non-degenerate hyperbolic surface is a hyperboloid, $\mathcal{H}$, given, in homogeneous coordinates, by the equation

$$x_0 x_3 = x_1 x_2. \tag{1.1}$$

Every hyperbolic quadric in $\mathbb{RP}^3$ is equivalent to $\mathcal{H}$ with respect to the action of the projective group $\mathrm{PGL}(4,\mathbb{R})$. Given a generic perturbation of $\mathcal{H}$ given by a smooth periodic function $f(u,v)$, we will prove that the quadratic points are the points (u,v) for which

$$\left\{ \begin{array}{rcl} f_{uuu} + f_u & = & 0, \\ f_{vvv} + f_v & = & 0. \end{array} \right. \tag{1.2}$$

The Sturm-Hurwitz theorem states that a smooth periodic function has no fewer zeroes than its first non-trivial harmonic. In particular, the equation $f'''(x) + f'(x) = 0$ has at least four distinct roots on the circle $[0, 2\pi)$ for every 2π-periodic function $f(x)$. This result implies the classical four-vertex theorem. In a recent paper [13] the following conjecture of V. Arnold is proved: if a plane wave front is Legendrian isotopic to a circle then it has at least four vertices. The vertices correspond to the solutions of the system

$$\left\{ \begin{array}{rcl} F_{uuu} + F_u & = & 0, \\ F_v & = & 0 \end{array} \right.$$

where $F(u, v)$ is a generating function of the corresponding Legendrian curve in the space of cooriented contact elements of the plane (contactomorphic to the jet space J^1S^1); $u \in S^1$ is a cyclic coordinate and $v \in \mathbb{R}^k$ is an auxiliary variable. One cannot help noticing that the above system bears a strong resemblance of our system (1.2); we believe that both are particular cases of multidimensional Sturm theory yet to be discovered.

In general, we do not know how to estimate below the number of solutions of (1.2). We will restrict ourselves to the case where f is a trigonometric polynomial of bidegree $\leq (2, 2)$ and prove a number of partial results. The geometric meaning of trigonometric polynomials of bidegree $\leq (2, 2)$ is that this class of functions f describes the perturbations of the hyperboloid that lie on a *quartic*. This way our considerations are related to an interesting problem of real algebraic geometry: study quadratic points on quartics.[2] It is worth mentioning that the topology of a quartic that has a $\mathbb{T}^2$-component C^∞-close to the standard hyperboloid is known. Such a quartic can have two components diffeomorphic to $\mathbb{T}^2$, or one $\mathbb{T}^2$-component with n spheres S^2, where $n = 0, 1, \ldots, 9$, see [6] for details.

Based on our partial results we conjecture that (1.2) has at least 8 distinct zeros. This would imply that a small perturbation of the hyperboloid has no less than 8 distinct quadratic points. Let us make a bolder conjecture: *every closed hyperbolic surface in* $\mathbb{RP}^3$ *has no less than 8 distinct quadratic points.* This conjecture is in the same relation to the Carathéodory conjecture as the six-vertex theorem to the four-vertex one.

2 Local analysis

In this section we formulate our problem and study the local invariants of hyperbolic surfaces and quadratic points.

[2]For (non-degenerate) cubic surfaces the situation is well understood. The quadratic points in this case are precisely the intersection points of the lines that lie on the surface (there are 27 complex lines, but not all of them must be real), see [15, 11]. In particular, if a cubic surface is diffeomorphic to $\mathbb{RP}^2$, then it has exactly 3 quadratic points [11].

2.1 Non-degenerate surface and quadratic points: definitions and simple properties

We collect here simple facts about quadratic points. Most of them are known and can be found in classical books, see [14, 7, 9].

Identify locally $\mathbb{RP}^3$ with the Euclidean space $\mathbb{R}^3$ with coordinates

$$x = x_1/x_0, \quad y = x_2/x_0, \quad z = x_3/x_0. \tag{2.1}$$

Given a hyperbolic surface M, these coordinates can be chosen in such a way that in a neighbourhood of a point m this surface is given by

$$z = xy + \tfrac{1}{3}\,(a\,x^3 + b\,y^3) + \tfrac{1}{2}\,(c\,x^2y + d\,xy^2) + O(4) \tag{2.2}$$

where a, b, c, d are some constants. Indeed, it suffices to chose the asymptotic directions at m as the x- and y-axes.

It is important to notice that the parameters a, b, c, d are *not* well-defined functions of m. These parameters depend on the choice of coordinates, for instance, the coordinate changes $(x, y) \mapsto (tx, t^{-1}y)$ with arbitrary t preserve the form of quation (2.2) but vary the parameters. Even the signs of a and b change as one changes (x, y) to $(-x, -y)$. The geometric meaning of a and b will be explained in Appendix, see also [10].

Nevertheless, zero sets $a = 0$ and $b = 0$ are well defined.

Fact 2.1. *Point m is a quadratic point if and only if the paramaters a and b in (2.2) vanish at m:*

$$\left\{ \begin{array}{lcl} a & = & 0 \\ b & = & 0. \end{array} \right. \tag{2.3}$$

Proof. First we check that the condition (2.3) is independent of the choice of coordinates x and y.

An osculating quadrics at a point m is as follows

$$z = xy + \tfrac{1}{2}\,(\gamma\,xz + \delta\,yz + \varepsilon\,z^2)\,, \tag{2.4}$$

where γ, δ and ε are arbitrary constants. Indeed, formula (2.4) defines the quadrics approximating M up to the terms of order 2. Let now m be a quadratic point. A quadric (2.4) is *hyperosculating* if it coincides with M up to the terms of order 3. This is the case if and only if $\gamma = c, \delta = d$ and the constants a and b in (2.2) vanish. $\Box$

Let us summarize the above calculations.

Fact 2.2. *(i) At a generic point, there exists a 3-parameter family of osculating quadrics given by formula (2.4).*

(ii) At a quadratic point, there is a 1-parameter family of hyperosculating quadrics.

Indeed, ε in (2.4) remains a free parameter.

We can now define explicitly the notion of a generic surface that will be essential for the sequel. The following definition is open and dense in C^∞-topology.

Definition 2.3. A non-degenerate hyperbolic surface M in $\mathbb{RP}^3$ is said to be generic, or in general position, if:
1) the sets $(a = 0)$ and $(b = 0)$ are smooth embedded curves in M with transversal intersections;
2) at each intersection point $(a = 0) \cap (b = 0)$ both curves $(a = 0)$ and $(b = 0)$ are transversal to the asimptotic directions.

We arrive at the following observation that justifies the formulation of our main problem.

Fact 2.4. *Quadratic points on a generic hyperbolic surface in* $\mathbb{RP}^3$ *are isolated.*

Indeed, condition (2.3) is of codimension 2 since the parameters a and b are two independent functions in (x, y).

A hyperbolic surface is a quadric if and only if it contains its asymptotic tangent lines at any point, cf. [17]. The next statement is nothing else but an infinitesimal version of this statement (see, e.g., [7], p. 62).

Fact 2.5. *Quadratic points are those points at which both asymptotic lines have inflections.*

The curves $(a = 0)$ and $(b = 0)$ on M are precisely the sets of inflection points of the two asymptotic foliations; the quadratic points are the intersection points $(a = 0) \cap (b = 0)$.[3]

Remark 2.6. For an arbitrary oriented smooth foliation on $\mathbb{T}^2$, the average curvature of the leaves with respect to the standard flat metric is zero, see [5]. Therefore the leaves of any foliation have inflection points. It is easy to find two transversal foliations with no points at which both leaves have inflections, see figure 2. This would be a counterexample to our conjecture if

[3]The union $(a = 0) \cup (b = 0)$ is usually called the *flecnodal curve*.

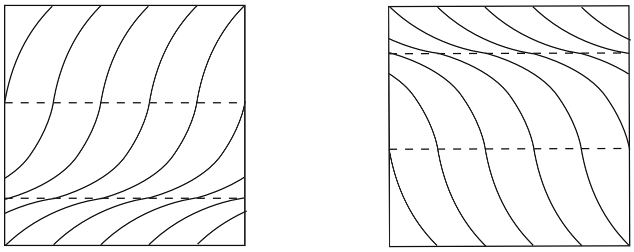

Figure 2: Transversal foliations with no common inflections.

one could realise these foliations as asymptotic lines on a hyperbolic surface.

2.2 Signature of a quadratic point

Recall that we consider only non-degenerate quadratic points. In this section we define an invariant of such a quadratic point that we call *signature*.

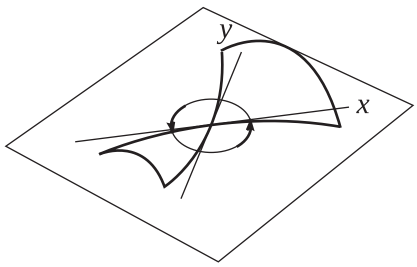

Figure 3: Natural ordering of x- and y-axis.

Fix an orientation of M and of $\mathbb{RP}^3$; the surface M is then co-oriented. The asymptotic x- and y- directions are naturally *ordered* at any point m. Indeed, choose z-coordinate in (2.2) positively coorienting M, consider the tangent plane T_mM and draw a small circle on it centered at m. Choose a point on the circle which lies above M and move in the positive direction; the first intersection with M corresponds to the x-axis, see figure 3.

At a quadratic point m, the surface M can "cross" the tangent plane in four different ways, see figure 4.

Definition 2.7. Define the signature $s = (s_1, s_2)$, where $s_i = +$ or $-$, of a quadratic point m. We put $s_1 = +$ (resp. $s_2 = +$) if the x-axis (resp. y-axis) in the vicinity of m lies under M. We put the $-$ sign otherwise.

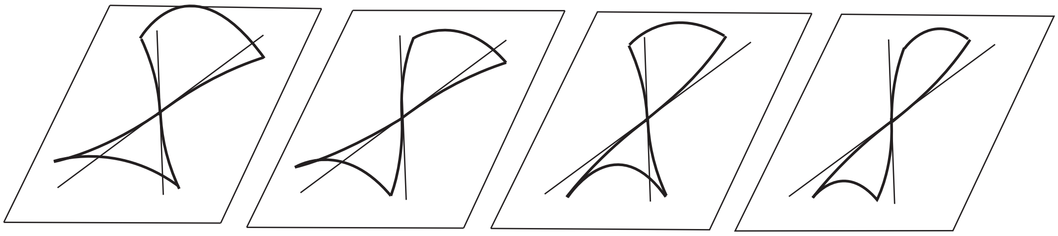

Figure 4: Quadratic points of signature $(+,+)$, $(+,-)$, $(-,+)$ and $(-,-)$.

Clearly, the signature is a $\mathrm{PGL}(4,\mathbb{R})$-invariant of a quadratic point. Note that if one changes the orientation of M or $\mathbb{RP}^3$, then the signs s_1 and s_2 change: $(+,+) \leftrightarrow (-,-)$ and $(+,-) \leftrightarrow (-,+)$. One can call the points of the two above types *even* and *odd*, respectively. This notion of parity is independent of the choice of orientation.

For every quadratic point m, coefficients a and b from (2.2) vanish at m. Consider the expansion (2.2) for a point close to m.

Lemma 2.8. *One has $s_1 = +$ if and only if $ax \geq 0$ on the x-axis; and $s_2 = +$ if and only if $by \geq 0$ on the y-axis.*

Proof. First notice that the coordinate change $(x,y) \mapsto (-x,-y)$ changes the signs of a and x simultaneously (as well as signs of b and y), so that the signs of the expressions $a\,x$ and $b\,y$ are well defined.

Consider the expansion (2.2) on the positive x-semiaxis. Since $y = 0$, one has $z(x,0) = \frac{1}{3}\,a\,x^3 + O(4)$. By definition of signature, $s_1 = +$ means that $z(x,0) > 0$. The curve $(a = 0)$ is transversal to the x-axis, and the statement follows. $\Box$

A family of non-degenerate hyperbolic surfaces M_t smoothly depending on a parameter $t \in [0,1]$ is called a *homotopy*. We do not assume *a-priori* that at each moment t the surface M_t is generic.

Quadratic points can be “created” or “annihilated” by homotopy in pairs, see figure 5.

Proposition 2.9. *Two quadratic points created/annihilated by a homotopy are of the same signature.*

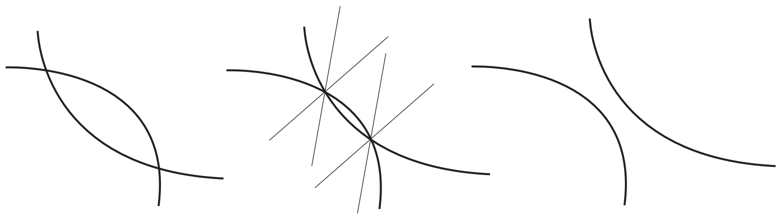

Figure 5: Creation/annihilation of quadratic ponits.

Proof. Close to the moment of creation/annihilation of a pair of quadratic points the curves $(a = 0)$ and $(b = 0)$ are transversal to both asymptotic directions. The statement then follows from Lemma 2.8. $\square$

Consider now a homotopy in the class of generic surfaces, i.e., M_t is generic for all $t \in [0, 1]$. Let us call such a homotopy *stable*. Signature is preserved by a stable homotopy.

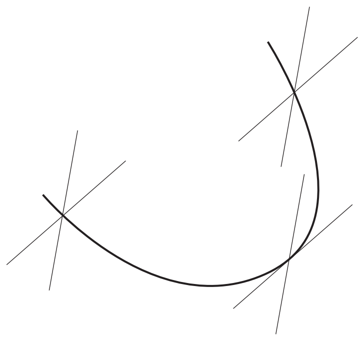

Figure 6: Curves $(a = 0)$ becomes non-transversal to the x-axis.

Lemma 2.10. *If a quadratic point $m \in M$ is not annihilated by a stable homotopy M_t, then the signature of m_t does not depent on t.*

Proof. Suppose that the homotopy connects points m_1 and m_2 of signature $(+, +)$ and $(-, +)$, respectively. Then there is a moment t_0 at which the curve $(a = 0)$ is not transversal to the x-axis, see figure 6. This contradicts to the fact that M_{t_0} is in general position. $\square$

2.3 Normal forms and differential invariants

The normal form of a non-degenerate hyperbolic surface M in the vicinity of a generic point m is one of the most classical results of projective differential geometry that goes back to Tresse and Wilczynski. In this section we give a much simpler proof of Tresse-Wilczynski's result; we then calculate the normal form (up to the order 4) in the vicinity of a point that lies on the curve $(a = 0)$ but not on $(b = 0)$, and finally in the vicinity of a quadratic point.

Normal forms are also discussed in [12] and [8], but the formula $\Pi_{3,1}$ of the former paper differs from the Tresse-Wilczynski's result, while the formula $\Pi_{4,3}$ is different from our formula (2.7) below.

Theorem 1. *Modulo projective transformations, a non-degenerate hyperbolic surface M is given, in a vicinity of a point m, by the formulas:*

(i) if m is generic then

$$z = xy + \tfrac{1}{3}\,(x^3 + y^3) + \tfrac{1}{12}\,(I\,x^4 + J\,y^4) + O(5), \tag{2.5}$$

(ii) if m belongs to the curve $(a = 0)$ but not to $(b = 0)$ then

$$z = xy + \tfrac{1}{3}\,(y^3 \pm x^3 y) + \tfrac{1}{12}\,\tilde{I}\,x^4 + O(5), \tag{2.6}$$

(iii) if m is quadratic then

$$z = xy \pm \tfrac{1}{3}\,(x^3\,y \pm x\,y^3) + \tfrac{1}{12}\left(\bar{I}\,x^4 + \bar{J}\,y^4\right) + O(5), \tag{2.7}$$

(all four combinations of signs are possible), where the parameters (I, J) are $\mathrm{PGL}(4, \mathbb{R})$*-invariants, as well as $\tilde{I}$ defined up to the sign; $(\bar{I}, \bar{J})$ are invariants defined up to the simultaneous sing change. If M is oriented then the signs of $\tilde{I}$ and $(\bar{I}, \bar{J})$ are well-defined.*

Proof. Consider the action of the Lie algebra $\mathrm{sl}(4, \mathbb{R})$ which is the infinitesimal version of the $\mathrm{PGL}(4, \mathbb{R})$-action. In affine coordinates (x, y, z), this action is spanned by 3 constant vector fields of "translations", together with 9 linear vector fields and 3 quadratic vector fields of "inversions":

$$\mathrm{sl}(4, \mathbb{R}) = \left\langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}, \quad x\,\tfrac{\partial}{\partial x}, \ldots, z\,\tfrac{\partial}{\partial z}, \quad x\,\mathcal{E}, y\,\mathcal{E}, z\,\mathcal{E} \right\rangle, \tag{2.8}$$

where $\mathcal{E} = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} + z\,\frac{\partial}{\partial z}$ is the Euler field.

Consider a hyperbolic surface M

$$z = xy + O(3)$$

where $O(3)$ stands for functions of x and y that belong to the cube of the maximal ideal (x, y). One readily checks the following

Lemma 2.11. *The Lie algebra of vector fields preserving the second jet of M is the subalgebra of dimension 7 spanned by*

$$\left\langle x\tfrac{\partial}{\partial x} + z\tfrac{\partial}{\partial z},\, y\tfrac{\partial}{\partial y} + z\tfrac{\partial}{\partial z}, \quad z\tfrac{\partial}{\partial x},\, z\tfrac{\partial}{\partial y}, \quad x\,\mathcal{E},\, y\,\mathcal{E},\, z\,\mathcal{E} \right\rangle. \tag{2.9}$$

Consider the expansion (2.2) and let us study the action of the Lie algebra (2.9) on the coefficients (a, b, c, d). The action of $X = \lambda\, x\,\mathcal{E} + \mu\, y\,\mathcal{E}$ is:

$$\dot{a} = 0, \quad \dot{b} = 0, \quad \dot{c} = 2\lambda, \quad \dot{d} = 2\mu,$$

where $\dot{}$ stands for the Lie derivative L_X; it follows that the flow of such an element "kills" the coefficients c and d.

The action of $X = \nu\,(x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}) + \kappa\,(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})$ is:

$$\dot{a} = 3\nu a, \quad \dot{b} = 3\kappa b, \quad \dot{c} = 2\nu c, \quad \dot{d} = 2\kappa d.$$

One concludes that the expansion of M can be reduced to the form

$$z = xy + \tfrac{1}{3}\,(x^3 + y^3) + O(4) \tag{2.10}$$

if m is generic so that $a \neq 0$, $b \neq 0$; to

$$z = xy + \tfrac{1}{3}\,y^3 + O(4), \tag{2.11}$$

if m belongs to the curve $(a = 0)$ but with $b \neq 0$, and to

$$z = xy + O(4), \tag{2.12}$$

if m is quadratic. Indeed, one can assume, without loss of generality, that $a \geq 0$ and $b \geq 0$ (it suffices to change the coordinates (x, y, z) to $(-x, y, -z)$, or $(x, -y, -z)$, or $(-x, -y, z)$ to change the signs of a and b). One then finds a vector field from (2.9) whose flow reduces the coefficients to a, b to 1; whenever they are different from 0.

Part (i). If m is generic then the subalgebra of (2.9) preserving the third-order expansion (2.10) is of dimensionl 3 and spanned by

$$\langle z\,\frac{\partial}{\partial x} - y\,\mathcal{E},\; z\,\frac{\partial}{\partial y} - x\,\mathcal{E},\; z\,\mathcal{E}\rangle,$$

Consider an arbitrary 4-th order expression

$$Q_4(x,y) = \alpha\,x^4 + \beta\,x^3y + \gamma\,x^2y^2 + \delta\,xy^3 + \varepsilon\,y^4, \tag{2.13}$$

the action of $X = \lambda\,(z\,\frac{\partial}{\partial x} - y\,\mathcal{E}) + \mu\,(z\,\frac{\partial}{\partial y} - x\,\mathcal{E}) + \nu\,z\,\mathcal{E}$ is given by

$$\dot\alpha = -\tfrac{1}{3}\,\mu, \quad \dot\beta = \tfrac{2}{3}\,\mu, \quad \dot\gamma = \nu, \quad \dot\delta = \tfrac{2}{3}\,\lambda, \quad \dot\varepsilon = -\tfrac{1}{3}\,\lambda,$$

so that one can kill the coefficients β, γ and δ. It follows that (2.5) is the normal form of M in a neighbourhood of a generic point.

Part (ii). The subalgebra of (2.9) preserving (2.11) is spanned by 4 vector fields

$$\langle \tfrac{2}{3}\,x\frac{\partial}{\partial x} + \tfrac{1}{3}\,y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},\; z\,\frac{\partial}{\partial x} - y\,\mathcal{E},\; z\,\frac{\partial}{\partial y} - x\,\mathcal{E},\; z\,\mathcal{E}\rangle.$$

As above, the action of $X = z\,\mathcal{E}$ allows one to kill the coefficient γ in (2.13). The action of $X = \lambda(z\,\frac{\partial}{\partial x} - y\,\mathcal{E}) + \mu(z\,\frac{\partial}{\partial y} - x\,\mathcal{E})$ on the 4-th order part reads

$$\dot\alpha = 0, \quad \dot\beta = 0, \quad \dot\gamma = 0, \quad \dot\delta = \tfrac{2}{3}\,\mu, \quad \dot\varepsilon = -\tfrac{1}{3}\,\lambda$$

that kills the coefficients δ and ε in (2.13). Finally, the action of the vector field $X = \frac{2}{3}\,x\frac{\partial}{\partial x} + \frac{1}{3}\,y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ is

$$\dot\alpha = \tfrac{5}{3}\,\alpha, \quad \dot\beta = \tfrac{4}{3}\,\beta, \quad \dot\gamma = \gamma, \quad \dot\delta = \tfrac{2}{3}\,\delta, \quad \dot\varepsilon = \tfrac{1}{3}\,\varepsilon,$$

so that the coefficient β can be reduced to ± 1. Formula (2.6) is proved. Simultaneous change of the signs: $(y,z) \leftrightarrow (-y,-z)$ changes the sign of $\tilde{I}$. This, of course, changes the (co)orientation of M defined by the z-axis.

Part (iii). The subalgebra of (2.9) preserving the third-order expansion (2.12) is spanned by 5 vector fields

$$\langle x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z},\; y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},\; z\,\frac{\partial}{\partial x} - y\,\mathcal{E},\; z\,\frac{\partial}{\partial y} - x\,\mathcal{E},\; z\,\mathcal{E}\rangle.$$

The action of $X = \lambda\,(x\,\frac{\partial}{\partial x} + z\,\frac{\partial}{\partial z}) + \mu\,(y\,\frac{\partial}{\partial y} + z\,\frac{\partial}{\partial z})$ is

$$\dot\alpha = (3\lambda - \mu)\,\alpha, \quad \dot\beta = 2\lambda\beta, \quad \dot\gamma = (\lambda + \mu)\,\gamma, \quad \dot\delta = 2\mu\delta, \quad \dot\varepsilon = (3\mu - \lambda)\,\varepsilon,$$

so that one can reduce β and δ to ± 1 if only these coefficients are different from zero, and this is the case since the quadratic point is generic . As above, one reduces the coefficient γ in (2.13) to 0. The actions of the fields $z\,\frac{\partial}{\partial x} - y\,\mathcal{E}$ and $z\,\frac{\partial}{\partial y} - x\,\mathcal{E}$ is trivial. Formula (2.7) follows.

Again, changing the signs: $(x, z) \leftrightarrow (-x, -z)$ or $(y, z) \leftrightarrow (-y, -z)$, one changes the signs $(\bar{I}, \bar{J}) \leftrightarrow (-\bar{I}, -\bar{J})$. This simultaneous sign change corresponds to the change of the orientation.

Theorem 1 is proved. $\Box$

Remark 2.12. Formula (2.5) is precisely the normal form of Tresse and Wilczynski (see [18], Second Memoir, formula (96)). The coefficients I, J and all of the following coefficients are called *absolute invariants*[4] of M.

Lemma 2.13. *The signature of a quadratic point is nothing else but the sign of the invariants in (2.7), namely* $(\sigma_1, \sigma_2) = (\mathrm{sgn}\bar{I}, \mathrm{sgn}\bar{J})$.

Proof. This follows directly from Definition 2.7. Indeed, restricting the right hand side of (2.7) to the x-axis, one has: $z = \bar{I}\,x^4 + O(5)$. Hence $\bar{I} > 0$ if and only if M is above the x-axis, and likewise for the y-axis. $\Box$

3 Small perturbations of the hyperboloid: linearization of the problem

In this section we deduce system (1.2) as the first-order approximation to our problem.

A *perturbation* of the hyperboloid $\mathcal{H}$ is a homotopy M_ε, smoothly depending on a small parameter $\varepsilon \in \mathbb{R}$, such that $M_0 = \mathcal{H}$. When we talk of "sufficiently small" perturbations, this means that there exists $\varepsilon_0 > 0$ such that the property we consider holds for all $|\varepsilon| \leq \varepsilon_0$.

The hyperboloid $\mathcal{H}$ defined by formula (1.1) has the following natural

[4]In Fifth Memoir Wilczynski develop the series up to order 6 and interpret the next 13 coefficients.

parametrization:

$$\begin{array}{rcl} x_0(u,v) & = & \cos\frac{u}{2}\,\cos\frac{v}{2}, \\ x_1(u,v) & = & \cos\frac{u}{2}\,\sin\frac{v}{2}, \\ x_2(u,v) & = & \sin\frac{u}{2}\,\cos\frac{v}{2}, \\ x_3(u,v) & = & \sin\frac{u}{2}\,\sin\frac{v}{2}, \end{array} \tag{3.1}$$

where $(u,v) \in [0,2\pi)$. The coordinates (u,v) on $\mathcal{H}$ are globally defined. Although $x_i(u,v)$ are not well defined functions on the torus, formula (3.1) gives a well-defined embedding $\mathbb{T}^2 \hookrightarrow \mathbb{RP}^3$.

We can describe a small perturbation of $\mathcal{H}$ in terms of a function on $\mathbb{T}^2$; the construction is as follows. The "normal" vector $X_{uv} := \frac{\partial^2}{\partial u \partial v}\,X(u,v)$, given more explicitly by

$$X_{uv} = \frac{1}{4}\left(\sin\frac{u}{2}\,\sin\frac{v}{2}, \quad -\sin\frac{u}{2}\,\cos\frac{v}{2}, \quad -\cos\frac{u}{2}\,\sin\frac{v}{2}, \quad \cos\frac{u}{2}\,\cos\frac{v}{2}\right), \tag{3.2}$$

is always transversal to $\mathcal{H}$, and so the family of surfaces

$$\widetilde{X}(u,v) = X(u,v) + \varepsilon\, f(u,v)\, X_{uv}, \tag{3.3}$$

where $f : \mathbb{T}^2 \to \mathbb{R}$ is an arbitrary smooth function, remains smooth for sufficiently small ε. Conversely, every surface M sufficiently close to $\mathcal{H}$ can be represented in a parametrized form by (3.3).

Proposition 3.1. *The perturbed surface (3.3) remains a quadric, in the first order in ε, if and only if the function f is a combination of the first harmonics:*

$$f = \sum_{-1 \leq n,m \leq 1} f_{n,m}\, e^{i(n\,u+m\,v)}, \tag{3.4}$$

where $f_{n,m} \in \mathbb{C}$ and $f_{n,m} = \overline{f}_{-n,-m}$ (since f is real).

Proof. From (3.1) and (3.2) one readily obtains the equation of the perturbed surface (3.3):

$$\widetilde{x}_0\widetilde{x}_3 - \widetilde{x}_1\widetilde{x}_2 = \frac{\varepsilon}{4}\, f + O(\varepsilon^2). \tag{3.5}$$

We will need the following

Lemma 3.2. *Function f is a combination of the first harmonics if and only if f is a quadratic expression in the coordinates $x_i(u,v)$ given by (3.1):*

$$f(u,v) = \sum \alpha_{ij}\, x_i\, x_j,$$

where α_{ij} are arbitrary constants.

Proof. The proof of this lemma is quite obvious. For instance, one has

$$\sin u \, \sin v = 4 \, \cos\tfrac{u}{2} \, \sin\tfrac{u}{2} \, \cos\tfrac{v}{2} \, \sin\tfrac{v}{2} = 4 \, x_0 \, x_4$$

and similarly for other homogeneous first-order harmonics, whereas

$$\sin u = 2 \, \cos\tfrac{u}{2} \, \sin\tfrac{u}{2} \left(\cos^2\tfrac{v}{2} + \sin^2\tfrac{v}{2}\right) = 2 \left(x_0 \, x_2 + x_1 \, x_4\right)$$

and similarly for other homogeneous harmonics of order $(0,1)$ or $(1,0)$, and finally one has

$$1 = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2$$

for the constant function. Lemma 3.2 follows. □

Equation (3.5) implies now that the perturbed surface (3.3) satisfies the quadratic equation:

$$\widetilde{x}_0\widetilde{x}_3 - \widetilde{x}_1\widetilde{x}_2 - \frac{\varepsilon}{4} \sum_{0 \leq i,j \leq 3} \alpha_{ij} \, \widetilde{x}_i\widetilde{x}_j + O(\varepsilon^2) = 0.$$

Therefore, the perturbed surface (3.3) remains quadric in the first order in ε. Proposition 3.1 is proved. □

Notice that the space of functions (3.4) is precisely the 9-dimensional space of solutions of the system

$$\left\{ \begin{array}{rcl} f_{uuu} + f_u & \equiv & 0, \\ f_{vvv} + f_v & \equiv & 0. \end{array} \right.$$

The following statement can be understood as a version of Proposition 3.1, but at a single point.

Theorem 2. *Given a perturbation (3.3) of the standard hyperboloid, a point with coordinates $(u,v) = (u_0, v_0)$ remains quadratic in the first order in ε if and only if condition (1.2) is satisfied at (u_0, v_0).*

Proof. Without loss of gererality consider the point with coordinates $(u_0, v_0) = (0,0)$. Identify locally $\mathbb{RP}^3$ and the Euclidean space $\mathbb{R}^3$ with the coordinates (2.1); the parametrized hyperboloid $\mathcal{H}$ is then given by

$$\begin{array}{rcl} x & = & \tan\frac{v}{2}, \\ y & = & \tan\frac{u}{2}, \\ z & = & \tan\frac{u}{2} \, \tan\frac{v}{2}. \end{array} \qquad (3.6)$$

Let us calculate the perturbed surface (3.3). One obtains

$$\widetilde{x} = \frac{\cos\frac{u}{2}\,\sin\frac{v}{2} - \frac{\varepsilon}{4}\,f\,\sin\frac{u}{2}\,\cos\frac{v}{2}}{\cos\frac{u}{2}\,\cos\frac{v}{2} + \frac{\varepsilon}{4}\,f\,\sin\frac{u}{2}\,\sin\frac{v}{2}}.$$

and similarly for $\widetilde{y}$ and $\widetilde{z}$. Finally one has

$$\begin{array}{rcl} \widetilde{x} & = & x - \frac{\varepsilon}{4}\,f\,(y + xz) + O(\varepsilon^2) \\ \widetilde{y} & = & y - \frac{\varepsilon}{4}\,f\,(x + yz) + O(\varepsilon^2) \\ \widetilde{z} & = & xy + \frac{\varepsilon}{4}\,f\,(1 - z^2) + O(\varepsilon^2). \end{array} \tag{3.7}$$

Assume that the point $(\widetilde{x}, \widetilde{y}, \widetilde{z}) = (0,0,0)$ of the perturbed surface is quadratic. This means its coordinates have to satisfy a quadratic equation up to the terms of order ≤ 2 in ε and ≤ 4 in $(\widetilde{x}, \widetilde{y}, \widetilde{z})$, namely

$$\widetilde{z} - \widetilde{x}\widetilde{y} = \varepsilon\,(P(\widetilde{x}, \widetilde{y}, \widetilde{z}) + O(4)) + O(\varepsilon^2),$$

where P is a polynomial of degree ≤ 2.

Exactly as in the case of condition (2.3), the coefficients of x^3 and y^3 in the above equation are the obstructions to existence of such a polynomial P. Indeed, these coefficients are identically zero (up to order 1 in ε) in the right hand side. Let us calculate these coefficients for the left hand side of the above equality.

From the Taylor expansion one obtains, for the case of x^3, the following expression: $\left(\frac{1}{24}\,f_{xxx} + \frac{1}{4}\,f_x\right)\varepsilon$, where the derivatives are takes at the point $(0,0)$. In the same way, one gets $\left(\frac{1}{24}\,f_{yyy} + \frac{1}{4}\,f_y\right)\varepsilon$ for y^3. Therefore, one obtains the following system:

$$\left\{ \begin{array}{rcl} \frac{1}{6}\,f_{xxx} + f_x & = & 0 \\ \frac{1}{6}\,f_{yyy} + f_y & = & 0. \end{array} \right.$$

Furthermore, the chain rule, applied to (3.6), implies that at point $(0,0)$ one has:

$$f_x = 2\,f_v \quad \text{and} \quad f_{xxx} = 8\,f_{vvv} - 4\,f_v,$$

so that the above system is precisely the system (1.2). $\Box$

The following statement is an immediate consequence of Theorem 2 and of compactness of the 2-torus.

Corollary 3.3. *Given a deformation M_ε defined by (3.3) with sufficiently small ε and a generic function f, the number of quadratic points on M_ε coincides with the number of the points for which the system (1.2) is satisfied.*

Indeed, for a generic function f, the solution of (1.2) are simple (of multiplicity 1) and cannot be removed by a small perturbation.

4 Approximation by quartics: second harmonics

We will be interested in the perturbations of the hyperboloid $\mathcal{H}$ in the class of quartics. More precisely, we will be looking for C^∞-families M_ε of quartics that contain a smooth component diffeomorphic to $\mathbb{T}^2$ and coinciding with $\mathcal{H}$ for $\varepsilon = 0$.

According to Proposition 3.1, the space of first harmonics (3.4) corresponds to the perturbations of $\mathcal{H}$ inside the space of quadrics. It turns out that the space of second harmonics also has a nice algebraic geometry meaning.

Proposition 4.1. *A perturbation (3.3) satisfies a quartic equation, in the first order in ε, if and only if the function f is given by the formula*

$$f = \sum_{-2 \le n,m \le 2} f_{n,m}\, e^{i(n\,u + m\,v)}, \tag{4.1}$$

where $f_{n,m} \in \mathbb{C}$ and $f_{n,m} = \overline{f}_{-n,-m}$.

Proof. Function f is as in (4.1) if and only if f can be written in terms of the coordinates (3.1) as a homogeneous quartic expression:

$$f = \sum_{0 \le i,j,k,\ell \le 3} \alpha_{ijk\ell}\, x_i x_j x_k x_\ell$$

where $\alpha_{ijk\ell}$ are some constants. The proof of this statement is similar to that of Lemma 3.2.

Equation (3.5) implies then that the perturbed surface (3.3) satisfies a homogeneous equation of order 4

$$(\widetilde{x}_0 \widetilde{x}_3 - \widetilde{x}_1 \widetilde{x}_2)\left(\widetilde{x}_0^2 + \widetilde{x}_1^2 + \widetilde{x}_2^2 + \widetilde{x}_3^2\right) - \frac{\varepsilon}{4} \sum_{0 \le i,j,k,\ell \le 3} \alpha_{ijk\ell}\, \widetilde{x}_i \widetilde{x}_j \widetilde{x}_k \widetilde{x}_\ell + O(\varepsilon^2) = 0,$$

and Proposition 4.1 follows. □

Let us calculate the dimension of the moduli space of quartic deformations of $\mathcal{H}$.

Proposition 4.2. *The space of* $\mathrm{PGL}(4,\mathbb{R})$*-classes of quartic deformations of* $\mathcal{H}$ *is of dimension 15.*

Proof. We give two ways to calculate the dimension of the space of deformations.

First. The space of second harmonics (4.1) is 25-dimensional. Its quotient by the space of first harmonics (that do not change $\mathcal{H}$ up to projective transformations, cf. Proposition 3.1) is 16. Finally, the quotient by homotheties $\mathbb{R}^*$ leaves us with a 15-dimensional space.

Second. The space $\mathbb{R}_4[x_0, x_1, x_2, x_3]$ of homogeneous polynomials of degree 4 is of dimension 35 (and so the dimension of the space of quartics is 34). The space of quartic deformations modulo the $\mathrm{PGL}(4,\mathbb{R})$-action is related to the quotient space $\mathbb{R}_4[x_0, x_1, x_2, x_3]/\mathcal{R}$, where $\mathcal{R}$ is the component of degree 4 of the ideal with two generators:

$$\mathcal{R} = \left\langle x_0x_3 - x_1x_2, \quad x_0^2 + x_1^2 + x_2^2 + x_3^2 \right\rangle .$$

One easily checks that $\dim \mathcal{R} = 19$, so that, taking into account the homotheties, we again obtain dimension 15. □

Naturally, both calculation yield the same answer, in accordance with Proposition 4.1.

5 Partial solutions to the main system

Unfortunately, we are unable to give a general estimate below of the number of solutions of system (1.2). We will consider the case where the function f belongs to the space of second harmonics, but even in this case our estimates are not complete. We will give two partial results and one example that we believe realizes the least number of solutions.

A. Consider first a 12-dimensional subspace of the space (4.1) with the condition $f_{2,2} = f_{2,-2} = f_{-2,2} = f_{-2,-2} = 0$, that is, the subspace of functions which are at most first harmonics in one of the variables.

Proposition 5.1. *If f is a generic function belonging to the above subspace, then there are at least 12 distinct points on the $[0, 2\pi) \times [0, 2\pi)$-torus at which system (1.2) is satisfied.*

Proof. One has:

$$
\begin{aligned}
f_{uuu} + f_u &= \phi_1(v) \cos 2u + \phi_2(v) \sin 2u, \\
f_{vvv} + f_v &= \psi_1(u) \cos 2v + \psi_2(u) \sin 2v
\end{aligned}
$$

where the functions ϕ_i and ψ_i belong to the space of first harmonics.

Consider first the curve $f_{vvv} + f_v = 0$ on $\mathbb{T}^2$. In the non-degenerate case (i.e., if all surfaces M_ε are in general position) this curve is of one of three free homotopy types: $2 \times (2, 1)$, or $2 \times (2, -1)$, or $4 \times (1, 0)$, see figure 7.

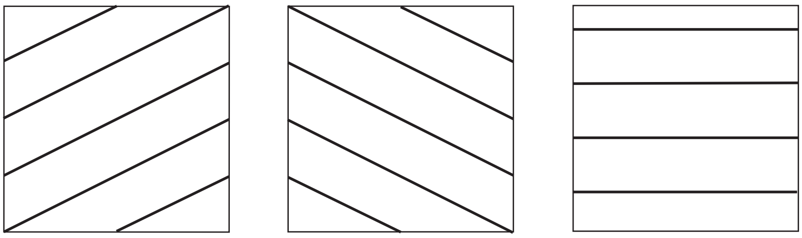

Figure 7: Curves on $\mathbb{T}^2$ of classes $2 \times (2, 1)$, $2 \times (2, -1)$ and $4 \times (1, 0)$.

Indeed, this curve intersects each "vertical" cycle $u = u_0$ in exactly 4 points, while it intersects each "horizontal" cycle $v = v_0$ in the same (even) number of points ≤ 2.

Similarly, the curve $f_{uuu} + f_u = 0$ is of one of three free homotopy types: $2 \times (1, 2)$, or $2 \times (-1, 2)$, or $4 \times (0, 1)$.

Since the number of intersection points of two curves of the free homotopy types $n \times (p, q)$ and $n' \times (p', q')$ is not less than $nn'|pq' - qp'|$, we conclude that, in our case, this number is at least 12. Indeed, this number is 12 for the curves $2 \times (2, 1)$ and $2 \times (1, 2)$, as well as for the curves $2 \times (2, -1)$ and $2 \times (-1, 2)$, it is equal to 16 in all the cases involving the curves $4 \times (0, 1)$ and $4 \times (1, 0)$, and it equals 20 for the intersection of the curves $2 \times (-1, 2)$ and $2 \times (2, -1)$. $\square$

Remark 5.2. It is easy to see how the signature of two "neighbouring" quadratic points changes, see figure Indeed, the intersecting curves in figure

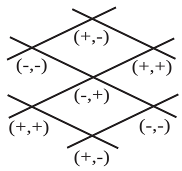

Figure 8: Signature changes.

8 are the curves where a and b change their signs. The statement then follows from Lemma 2.8.

B. Let us now consider the space of homogeneous second-order harmonics:

$$f = \cos 2u\,(\alpha_{11}\cos 2v + \alpha_{12}\sin 2v) + \sin 2u\,(\alpha_{21}\cos 2v + \alpha_{22}\sin 2v),$$

where α_{ij} are arbitrary constants. In this case, both curves, $f_{uuu} + f_u = 0$, and $f_{vvv} + f_v = 0$, on $\mathbb{T}^2$ are either of homological type $4 \times (1,1)$, see figure 9, or both of type $4 \times (-1,-1)$, and they may avoid intersecting each other altogether. Simple topological considerations as we used in the proof of Proposition 5.1, cannot be applied in this case. However, the curves $f_{uuu} + f_u = 0$, and $f_{vvv} + f_v = 0$ are no more independent.

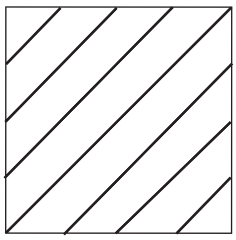

Figure 9: Curves on $\mathbb{T}^2$ of type $4 \times (1,1)$.

Proposition 5.3. *If f is a homogeneous second harmonic then there are at least 32 distinct points on the $[0,2\pi) \times [0,2\pi)$-torus at which system (1.2) is satisfied.*

Proof. It is straightforward to check that the system (1.2) is equivalent in this case to the following system:

$$\tau = \frac{\alpha_{11}\,t + \alpha_{12}}{\alpha_{21}\,t + \alpha_{22}}, \qquad \tau = \frac{\alpha_{22}\,t - \alpha_{21}}{-\alpha_{12}\,t + \alpha_{11}},$$

where $\tau = \tan 2u$ and $t = \tan 2v$. This system is $\frac{\pi}{2}$-periodic and leads to the quadratic equation

$$(\alpha_{11}\alpha_{12} + \alpha_{21}\alpha_{22})\, t^2 + (\alpha_{22}^2 - \alpha_{21}^2 + \alpha_{12}^2 - \alpha_{11}^2)\, t - (\alpha_{11}\alpha_{12} + \alpha_{21}\alpha_{22}) = 0,$$

whose descriminant is strictly positive. It follows that system (1.2) has exactly 2 solutions on $[0, \frac{\pi}{2}) \times [0, \frac{\pi}{2})$ and thus 32 solutions on $[0, 2\pi) \times [0, 2\pi)$. $\square$

Remark 5.4. Unlike the previous example, the signature of the "neighbour" quadratic points is the same, see figure 10. Indeed, these quadratic points are

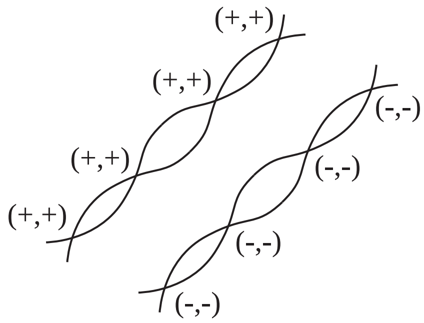

Figure 10: Signature in the homogeneous case.

the points of intersection of the same curves $a = 0$ and $b = 0$ which remain transversal to the asimptotic directions, so that a and b do not change their signs on the corresponding x- and y-axes, cf. Lemma 2.8.

C. Let us now give an example of a function f for which system (1.2) has 8 solutions on the $[0, 2\pi) \times [0, 2\pi)$-torus. We will consider a sum of functions of the two above classes.

Example 5.5. Let $f = \cos(2u - v) + \varepsilon \cos(2u - 2v)$. Then the curve $f_{uuu} + f_u = 0$ is of type $2 \times (2, 1)$, see figure 7, while the curve $f_{vvv} + f_v = 0$ is of type $4 \times (1, 1)$, see figure 9. For $\varepsilon = 0$, the curves intersect transversally, hence, for sufficiently small ε, the number of intersection points is the same as for $\varepsilon = 0$, that is, equals the number of solutions of the system

$$\sin(2u - 2v) = \sin(2u - v) = 0.$$

This number is equal to 8.

Appendix: Wilczynski system of equations

To provide a different description of hyperbolic surfaces in $\mathbb{RP}^3$ we will write down the system of differential equations introduced by E. Wilczynski [18][5] (we also refer to [10] for a modern exposition). Given a parameterized surface $x(u,v) \subset \mathbb{RP}^3$, one wants to lift it canonically into the vector space $\mathbb{R}^4$ equipped with the standard volume form.

Let us introduce the notion of *asymptotic coordinates* in a neighbourhood of an arbitrary point $m \in M$. These are coordinates (u,v) with origin at m such that the asymptotic lines on M are precisely the coordinate lines $u = \text{const}$ and $v = \text{const}$. Clearly, asymptotic coordinates are defined modulo the transformations $(u,v) \to (U(u), V(v))$. Let first $X(u,v) \subset \mathbb{R}^4$ be an arbitrary lift. The four vectors X, X_u, X_v, X_{uv} are linearly independent for every (u,v). One can uniquely fix the lift of the parameterized surface $x(u,v)$ into $\mathbb{R}^4$ by the condition

$$|X\, X_u\, X_v\, X_{uv}| = 1. \tag{5.1}$$

Let us call this lift *canonical*.

A straightforward calculation leads to the following fact. The coordinates of the canonical lift satisfy the system of linear differential equations

$$\begin{array}{rcl} X_{uu} + a\, X_v + \alpha\, X & = & 0 \\ X_{vv} + b\, X_u + \beta\, X & = & 0 \end{array} \tag{5.2}$$

where a, b, α, β are functions in (u,v) satisfying the integrability conditions

$$\begin{array}{rcl} \alpha_{vv} + b\,\alpha_u + 2b_u\,\alpha - \beta_{uu} - 2a_v\,\beta - a\,\beta_v & = & 0 \\ a\, b_v + 2a_v\, b + b_{uu} + 2\beta_u & = & 0 \\ b\, a_u + 2b_u\, a + a_{vv} + 2\alpha_v & = & 0. \end{array} \tag{5.3}$$

Conversely, system (5.2) whose coefficients satisfy relations (5.3) corresponds to a non-degenerate parameterized surface $M \subset \mathbb{RP}^3$.

System (5.2) is called the canonical (or the Wilczynski) system of differential equations associated with a surface in $\mathbb{RP}^3$.

Proposition 5.6. *The quadratic points on M are the points at which the functions $a(u,v)$ and $b(u,v)$ vanish.*

[5]This reference is a first systematic study of hyperbolic surfaces in $\mathbb{RP}^3$

Proof. Identify locally $\mathbb{RP}^3$ and $\mathbb{R}^3$ and consider an affine lift $X(u, v)$. The linear coordinates (x, y, z) in the affine 3-space can be chosen in such a way that $X(0, 0)$ is the origin and the vectors $X_u(0, 0), X_v(0, 0)$ and $X_{uv}(0, 0)$ are the coordinate vectors. Then the surface is locally given by the equation $z = xy + O(3)$ where $O(3)$ stands for terms, cubic in x, y. One then checks (see, e.g., [10]) that the equation defining M is

$$z = xy + \frac{1}{3} (a\, x^3 + b\, y^3) + O(4).$$

and then applies condition (2.3). □

For the sake of completeness let us clarify the geometric meaning of the coefficients a and b. The proof of the following statement is a straightforward calculation.

Proposition 5.7. *Under coordinate transformations $(u, v) \mapsto (U, V)$, the coefficients a and b transform as follows:*

$$a(u, v) \mapsto a(U, V)\, \frac{U_u^2}{V_v}, \qquad b(u, v) \mapsto b(U, V)\, \frac{V_v^2}{U_u}.$$

In other words, the following tensor fields

$$a = a(u, v)\, du^2 dv^{-1}, \qquad b = b(u, v)\, du^{-1} dv^2$$

are well defined. Further details can be found in [10], Section 5.1.

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